# Lesson 30 - Extrema of Functions of Two Variables II Applications

Last class, we learned the second partial derivatives test. We used this test to help us find local minima and local maxima of functions when possible. Today we will use this test to help us solve optimization problems.

## I Strategy for Optimization Problems

(1) **Read** the problem. Then **read it again** for details.

(2) Write your **variables** and what they mean.

(3) Write an **objective function** and whether you are trying to **maximize** it or **minimize** it.

(4) Write any **constraint equations** specified in the problem.

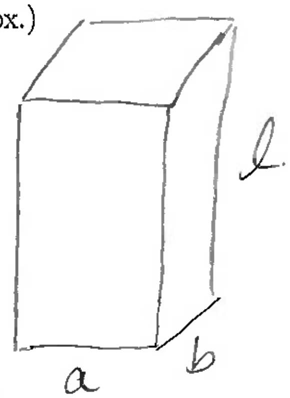
(5) If necessary, use the constraint equations to rewrite the **objective function** using only **two variables**.

(6) Use the **Second Partial Derivative Test** to find places where local minima or local maxima occur.

(7) **Answer the question.** Sometimes the question will be asking for places where there is an extremum. Sometimes they will want the extreme value. If they want the extreme value, you will need to plug your critical point into the objective function to answer the question.

## II Examples

**Example 1** (From OpenStax *Calculus Volume 3* textbook by Strange, Herman, et.al). The sum of the length and the girth (perimeter of a cross-section) of a package carried by a delivery service cannot exceed 108 in. Find the largest volume that can be sent. (**NOTE:** The length is the longest dimension of the box.)




max $V = lab$

Constraint: $l + 2a + 2b \leq 108$

Obviously max volume will satisfy

$$l + 2a + 2b = 108$$

$$\Rightarrow \quad l = 108 - 2a - 2b$$

So $V(a,b) = (108 - 2a - 2b)ab$

$$V(a,b) = 108ab - 2a^2 b - 2ab^2$$

$$V_a(a,b) = 108b - 4ab - 2b^2 = b(108 - 4a - 2b)$$

$$V_b(a,b) = 108a - 2a^2 - 4ab = a(108 - 2a - 4b)$$

$V_a(a,b) = 0 \quad \Rightarrow \quad b(108 - 4a - 2b) = 0$

$\cancel{b = 0}$ OR $108 - 4a - 2b = 0$

$4a + 2b = 108$

$a, b, l > 0$.

$V_b(a,b) = 0 \quad \Rightarrow \quad a(108 - 2a - 4b) = 0$

$\cancel{a = 0}$ OR $108 - 2a - 4b = 0$

$2a + 4b = 108$

$$\begin{cases} 4a + 2b = 108 & ① \\ 2a + 4b = 108 & ② \end{cases}$$

① $- 2 *$ ②

$$\begin{aligned} 4a + 2b &= 108 \\ -4a - 8b &= -216 \\ \hline -6b &= -108 \\ b &= 18 \end{aligned}$$

$$\begin{aligned} 4a + 2b &= 108 \\ 4a + 2(18) &= 108 \\ 4a &= 72 \\ a &= 18 \end{aligned}$$

Critical point

$(a, b) = (18, 18)$

$V_{aa}(a,b) = -4b$

$V_{bb}(a,b) = -4a$

$V_{ab}(a,b) = 108 - 4a - 4b$

$$\begin{aligned} D(18,18) &= \left(-4(18)\right)\left(-4(18)\right) - \left(108 - 4(18) - 4(18)\right)^2 \\ &= (-72)(-72) - (-36)^2 \\ &= 3888 > 0 \end{aligned}$$

(can you see that D is (+) without a calculator?)

$V_{aa}(18,18) = -4(18) < 0$

$\Rightarrow$ relative max @ $(18,18)$

relative max volume

$V(18,18) = (108 - 2(18) - 2(18))(18)(18)$

$= (36)(18)(18)$

$= 11{,}664 \text{ in}^3$

(If they had asked for the dimensions, then the answer would have been

18 in x 18 in x 36 in

length

**Example 2** (Based on LON-CAPA problem). A manufacturer is planning to sell a new product at the price of 300 dollars per unit and estimates that if $x$ thousand dollars is spent on development and $y$ thousand dollars is spent on promotion, consumers will buy approximately

$$\frac{800y}{y+5} + \frac{900x}{x+10}$$

units of the product.

If manufacturing costs for the product are 200 dollars per unit, how much should the manufacturer spend on development and how much on promotion to generate the largest possible profit?

Remember: Profit = All Revenues - All Expenses/Costs

Round your answer to the nearest dollar.

$$P(x,y) = 300\left(\frac{800y}{y+5} + \frac{900x}{x+10}\right) - 200\left(\frac{800y}{y+5} + \frac{900x}{x+10}\right)$$
$$-1000x - 1000y$$

$$P(x,y) = 100\left(\frac{800y}{y+5} + \frac{900x}{x+10}\right) - 1000x - 1000y.$$

$$P_x(x,y) = 100\left(\frac{(x+10)\cdot 900 - 900x}{(x+10)^2}\right) - 1000$$

$$= \frac{90000}{(x+10)^2} - 1000$$

$$= \frac{900,000 - 1,000(x+10)^2}{(x+10)^2}$$

$$P_x(x,y) = 0 \Rightarrow 900,000 - 1,000(x+10)^2 = 0$$
$$1000(x+10)^2 = 900000$$
$$(x+10)^2 = 900$$
$$x+10 = \pm\sqrt{900}$$
$$x = -10 \pm 30$$
$$x = \cancel{-40} \text{ OR } 20$$

$x > 0$.

Similarly.

$$P_y(x,y) = 100\left(\frac{(y+5)\,800 - 800y}{(y+5)^2}\right) - 1000$$

$$= \frac{400{,}000}{(y+5)^2} - 1000$$

$$= \frac{400{,}000 - 1000(y+5)^2}{(y+5)^2}$$

$$P_y(x,y) = 0 \Rightarrow 400{,}000 - 1000(y+5)^2 = 0$$

$$(y+5)^2 = 400$$

$$y+5 = \pm\sqrt{400}$$

$$y = 5 \pm 20$$

$$y = \cancel{-15} \text{ OR } 25$$

$$(y \geq 0).$$

Critical point: $(x, y) = (20, 25)$

$$P_x(x,y) = 900{,}000(x+10)^{-2} \quad -1000$$

$$P_y(x,y) = 400{,}000(y+5)^{-2} \quad -1000$$

So

$$P_{xx}(x,y) = \frac{-1{,}800{,}000}{(x+10)^3}$$

$$P_{yy}(x,y) = \frac{-800{,}000}{(y+5)^3}$$

$$P_{xy}(x,y) = 0$$

$$D(20,25) = \left(\frac{-1{,}800{,}000}{(20+10)^3}\right)\left(\frac{-800{,}000}{(25+5)^3}\right) - 0^2 > 0.$$

$$P_{xx}(20,25) = \frac{-1{,}800{,}000}{(20+10)^3} < 0$$

$\Rightarrow$ Relative max @ $x = 20$, $y = 25$

Ans: Spend \$20,000 on development and \$25,000 on promotion.

**Example 3** (Based on LON-CAPA problem). A machine's productivity is based on the measurements of $x$ and $y$ in the room containing the machine. (Note that $x \geq 0$ and $y \geq 0$.) The machine's productivity is given by

$$f(x,y) = xye^{-\frac{x^2}{18}-\frac{y^2}{50}}$$

For product rule $\left\{ \begin{array}{l} xy \ - \text{1st} \\ e^{-\frac{x^2}{18}-\frac{y^2}{50}} \ - \text{2nd} \end{array} \right.$

(a) Find the critical points of $f$.

$$f_x(x,y) = \underbrace{xy}_{\text{1st}} \underbrace{e^{-\frac{x^2}{18}-\frac{y^2}{50}}\left(\frac{-2x}{18}\right)}_{(\text{2nd})'} + \underbrace{y}_{(\text{1st})'} \underbrace{e^{-\frac{x^2}{18}-\frac{y^2}{50}}}_{\text{2nd}}$$

$$= ye^{-\frac{x^2}{18}-\frac{y^2}{50}}\left(\frac{-x^2}{9}+1\right)$$

$f_x(x,y) = 0$ if $y = 0$ OR $\underbrace{e^{-\frac{x^2}{18}-\frac{y^2}{50}} = 0}_{\text{Never}}$ or $\frac{-x^2}{9}+1 = 0$

$x = 3.$

Similarly

$$f_y(x,y) = xe^{-\frac{x^2}{18}-\frac{y^2}{50}}\left(\frac{-y^2}{25}+1\right)$$

$f_x(x,y) = 0$ if $x = 0$ OR $y = 5$

Critical points occur when

$\left(\begin{array}{c} x=3 \\ \text{OR} \\ y=0 \end{array}\right)$ AND $\left(\begin{array}{c} x=0 \\ \text{OR} \\ y=5 \end{array}\right) \longrightarrow$ Only 2 critical points.

$(3,5)$ or $(0,0)$

(b) If $f_{xx}(3,5) = -\frac{10}{9e}$, $f_{yy} = -\frac{6}{5e}$, and $f_{xy}(3,5) = 0$, what can you conclude about the productivity of the machine when $x = 3$ and $y = 5$? Looking back at the original function, can you explain why $x = 0$ and $y = 0$ could not possibly be a useful solution if we want to maximize productivity?

$$\left.\begin{array}{l} D(3,5) = \left(\frac{-10}{9e}\right)\left(\frac{-6}{5e}\right) - 0^2 > 0 \\ f_{xx}(3,5) = \frac{-10}{9e} < 0 \end{array}\right\} \rightarrow$$ Relative max productivity when $x=3$ and $y=5$

$f(x,y) = xye^{(\quad)} > 0$ if $x > 0$ and $y > 0$

$f(0,0) = 0 \longrightarrow$ certainly not the max